1. Exercises from Sections 3.1

Review of the implicit function theorem

Let $F(x, y) : \mathbb{R}^{n+k} \to \mathbb{R}^k$ be a differentiable function (here $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_k)$). The implicit function theorem gives sufficient conditions for which we can *locally* express the surface $F(x, y) = 0$ by $y = f(x)$ for some differentiable map f. More colloquially, the theorem says that surfaces defined implicitly by the vanishing systems of equations look like the graphs of functions in the neighbourhood of most points. The precise statement is as follows:

THEOREM 1. *Let $F(x, y) : \mathbb{R}^{n+k} \to \mathbb{R}^k$ be a differentiable function and suppose we have a point $(a, b) \in \mathbb{R}^{n+k}$ such that $F(a, b) = 0$. If,*

$$\det\left(\frac{\partial F_i}{\partial y_j}\right)\bigg|_{(x,y)=(a,b)} \neq 0$$

then there exist numbers $R_1, R_2 > 0$ such that for all $x \in \{(x_1, \ldots, x_n) \,|\, |x - a| < R_1\} = U$ there exists a unique $y \in \{(y_1, \ldots, y_k) \,|\, |y - b| < R_2\} = V$ with $F(x, y) = 0$. Such a correspondence "implicitly" defines a function $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^k$ by the rule $f(x) = y$.

PROBLEM 1. *Can the equation $(x^2 + y^2 + 2z^2)^{1/2} = \cos z$ be solved uniquely for y in terms of x, z near $(0, 1, 0)$? For z in terms of x and y?*

Solution:

- Let $F(x, y, z) = (x^2 + y^2 + 2z^2)^{1/2} - \cos z$, then $F(0, 1, 0) = (0 + 1 + 0)^{1/2} - 1 = 0$.
- We can try to solve for y in terms of x and z implicitly using theorem 1.

$$\frac{\partial F}{\partial y}\bigg|_{(0,1,0)} = \left[\frac{y}{\sqrt{x^2 + y^2 + 2z^2}}\right]\bigg|_{(0,1,0)} = 1 \neq 0$$

- As $\partial_y F \neq 0$ at $(0, 1, 0)$, the implicit function theorem tells us that we can solve for y in terms of (x, z) near this point.
- Can we find a function $z = f(x, y)$ such that $F(x, y, f(x, y)) = 0$ near $(0, 1, 0)$?

$$\frac{\partial F}{\partial z} = \frac{2z}{\sqrt{x^2 + y^2 + 2z^2}} + \sin z$$

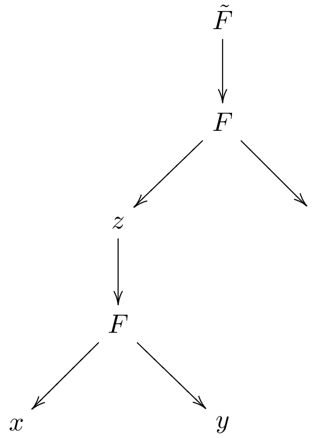
- $\partial_z F = 0$ at $(0, 1, 0)$, so we cannot say whether $F(x, y, z) = 0$ can be expressed as a graph $z = f(x, y)$.

PROBLEM 2. *Suppose $F(x, y) \in C^1$ and $F(0, 0) = 0$. What conditions on F will guarantee that $F(F(x, y), y) = 0$ can be solved for y as a C^1 function of x near $(0, 0)$?*

Solution:

- Let $\tilde{F}(x, y) = F(F(x, y), y)$.
- By the implicit function theorem, we can solve for y as a function of x if $\partial_y \tilde{F} \neq 0$

- We will use a placeholder $z = F(x, y)$



- Notice: $\partial_z F = \partial_x F|_{x=z}$
- By the chain rule,

$$\partial_y \tilde{F} = \partial_z F \partial_y F + \partial_y F = \partial_y F(\partial_x F + 1) \neq 0$$

- So we must have $\partial_y F \neq 0$ and $\partial_x F \neq -1$

Problem 3. *Consider the map: $F(x, y, z, u, v) : \mathbb{R}^5 \to \mathbb{R}^2$ given by:*

$$F(x, y, z, u, v) = \begin{pmatrix} xy^2 + xzu_1 + yu_2^2 - 3 \\ u_1^3 yz + 2xu_2 - u_1^2 u_2^2 - 2 \end{pmatrix}$$

And notice $F(1, 1, 1, 1, 1) = 0$. Can we solve for u, v as functions of x, y, z near $(1, 1, 1, 1, 1)$?

Solution:
Again, we are going to use the implicit function theorem.

$$\begin{aligned} \frac{\partial F_i}{\partial u_j} &= \begin{pmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} \end{pmatrix} \\ &= \begin{pmatrix} xz & 2yu_2 \\ 3u_1^2 yz - 2u_1 u_2^2 & 2x - 2u_1^2 u_2 \end{pmatrix} \end{aligned}$$

Which, evaluated at $(1, 1, 1, 1, 1)$ gives:

$$\frac{\partial F_i}{\partial u_j} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

Which has non-zero determinant. By the implicit function theorem, we may solve for u_1, u_2 in terms of (x, y, z) locally near $(1, 1, 1, 1, 1)$